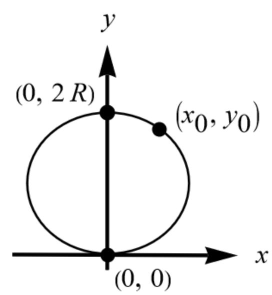
Exercise 8

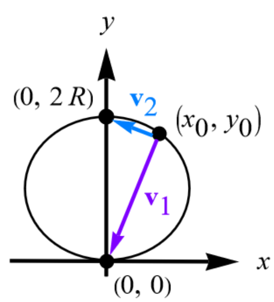
Suppose T is a triangle formed by placing three points on a circle, two of which lie on the circle's diameter. Use the previous problem to show T is a right triangle.

Solution

Suppose there are three points on a circle, two of which are on the diameter. Let this circle lie in the xy-plane so that the diameter is along the y-axis.



Draw the vector from (x_0, y_0) to $(0, 0)$ and the vector from (x_0, y_0) to $(0, 2R)$. Call these vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively.



$$\mathbf{v}_1 = (0, 0) - (x_0, y_0) = (-x_0, -y_0)$$

$$\mathbf{v}_2 = (0, 2R) - (x_0, y_0) = (-x_0, 2R - y_0)$$

The equation of the circle is $x^2 + (y - R)^2 = R^2$, so the point that isn't on the diameter satisfies

$$x_0^2 + (y_0 - R)^2 = R^2$$

$$x_0^2 + y_0^2 - 2y_0R + \cancel{R^2} = \cancel{R^2}$$

$$x_0^2 = 2y_0R - y_0^2$$

$$x_0 = \sqrt{y_0(2R - y_0)}.$$

As a result,

$$\mathbf{v}_1 = \left(-\sqrt{y_0(2R - y_0)}, -y_0\right)$$

$$\mathbf{v}_2 = \left(-\sqrt{y_0(2R - y_0)}, 2R - y_0\right).$$

Take the dot product of these two vectors.

$$\begin{aligned}\mathbf{v}_1 \cdot \mathbf{v}_2 &= \left[-\sqrt{y_0(2R - y_0)}\right]\left[-\sqrt{y_0(2R - y_0)}\right] + (-y_0)(2R - y_0) \\ &= y_0(2R - y_0) - y_0(2R - y_0) \\ &= 2Ry_0 - y_0^2 - 2Ry_0 + y_0^2 \\ &= 0\end{aligned}$$

This means that $\mathbf{v}_1$ and $\mathbf{v}_2$ are perpendicular and form a right triangle.